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A RATIONALITY CONDITION FOR THE EXISTENCE OF ODD PERFECT NUMBERS

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Abstract. A rationality condition is derived for the existence of odd perfect numbers involving the square root of a product, which consists of a sequence of repunits multiplied by twice the base of one of the repunits. This constraint also provides an upper bound for the density of odd integers which could satisfy $\frac{\sigma(N)}{N} = 2$, where N belongs to a fixed interval with a lower limit greater than 10^{300}. Characteristics of prime divisors of repunits are used to establish whether the product containing the repunits can be a perfect square. It is shown that the arithmetic primitive factors of the repunits with different prime bases can be equal only when the exponents are different, with possible exceptions derived from solutions of a prime equation. This equation is one example of a more general prime equation, $\frac{q_j^n - 1}{q_i^n - 1} = p^h$ and the demonstration of the non-existence of solutions when $h \geq 2$ requires the proof of a special case of Catalan's conjecture. Results concerning the exponents of prime divisors of the repunits are obtained, and they are combined with the method of induction to prove a general theorem on the non-existence of prime divisors satisfying the rationality condition.

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1. Introduction

The algorithm for demonstrating the non-existence of odd perfect numbers with fewer than nine different prime divisors requires the expansion of the ratio $\frac{\sigma(N)}{N}$ and strict inequalities imposed on the sums of powers of the reciprocal of each prime divisor [1][2]. Although it is possible to establish that $\frac{\sigma(N)}{N} \neq 2$ when N is divisible by certain primes, there are odd integers with a given number of prime divisors such that $\frac{\sigma(N)}{N} > 2$, while $\frac{\sigma(N)}{N} < 2$ for other integers with the same number of distinct prime factors. Moreover, the range of the inequality for $\left|\frac{\sigma(N)}{N} - 2\right|$ can be made very small even when N has a few prime factors. Examples of odd integers with only five distinct prime factors have been found that produce a ratio nearly equal to 2: $\left|\frac{\sigma(N)}{N} - 2\right| < 10^{-12}$ [3]. Since it becomes progressively more difficult to establish the inequalities as the number of prime factors increases, a proof by method of induction based on this algorithm cannot be easily constructed.

In §2, it will be shown that there is a rationality condition for the existence of odd perfect numbers. Setting $\frac{\sigma(N)}{N}$ equal to 2 is equivalent to equating the square root of a product, $2(4k+1)\prod_{i=1}^{\ell} \frac{q_i^{n_i}-1}{q_i-1} \frac{(4k+1)^{4m+2}-1}{4k}$, which contains a sequence of repunits, with a rational number. This relation provides both an upper bound for the density of odd perfect numbers in any fixed interval in $\mathbb{N}$ with a lower limit greater than 10^{300} and a direct analytical method for verifying their non-existence, since it is based on the irrationality of the square root of any unmatched prime divisors in the product. This condition is used in §3 to demonstrate the non-existence of a special category of odd perfect numbers. The properties of prime divisors of Lucas sequences required for the study of the square root of the product of the repunits are described in §4 and §5. An induction argument is constructed in §6, which proves that the square root expression is not rational for generic sets of prime divisors, each containing a large number of elements. This is first established for odd integers with four distinct prime divisors and then by induction using the properties of the divisors of the repunits.

2. Rationality Condition for the Existence of Odd Perfect Numbers

From the condition for the odd integer $N = (4k+1)^{4m+1}s^2 = (4k+1)^{4m+1}q_1^{2\alpha_1}...q_\ell^{2\alpha_\ell}$, $gcd(4k+1, s) = 1$ [4]-[6] to be a perfect number,

$$\frac{\sigma(N)}{N} = \left[\frac{(4k+1)^{4m+2}-1}{4k(4k+1)^{4m+1}}\right]\frac{\sigma(s^2)}{s^2} = \left[\frac{(4k+1)^{4m+2}-1}{4k(4k+1)^{4m+1}}\right]\left[\frac{\sigma(s)^2}{s^2}\right]\left[\frac{\sigma(s^2)}{\sigma(s)^2}\right] = 2 \quad (1)$$

it follows that

$$\frac{\sigma(s)}{s} = \sqrt{2}\prod_{i=1}^{\ell}\frac{(q_i^{\alpha+1}-1)}{(q_i - 1)^{\frac{1}{2}}(q_i^{2\alpha_i+1} - 1)^{\frac{1}{2}}} \times \left[\frac{4k(4k+1)^{4m+1}}{(4k+1)^{4m+2} - 1}\right]^{\frac{1}{2}} \tag{2}$$

and

$$\prod_{i=1}^{\ell}\frac{1}{(q_i^{\alpha_i+1} - 1)}\frac{\sigma(s)}{s} = \sqrt{2}\prod_{i=1}^{\ell}\frac{1}{(q_i^{2\alpha_i+1} - 1)^{\frac{1}{2}}(q_i - 1)^{\frac{1}{2}}} \times \left[\frac{4k(4k+1)^{4m+1}}{(4k+1)^{4m+2} - 1}\right]^{\frac{1}{2}} \tag{3}$$

Consistency of equation (3) for finite ℓ requires rationality of the entire square root expression.

The known integer solutions to $\frac{x^n - 1}{x - 1} = y^2$ [7]-[9] do not include the pairs $(x,n) = (4k+1, 4m+2)$, implying that $\left[\frac{(4k+1)^{4m+2} - 1}{4k}\right]^{\frac{1}{2}}$ is not a rational number. The number $[1+q_i+q_i^2+...+q_i^{2\alpha_i}]^{\frac{1}{2}}$ is only rational when $q_i = 3,\ \alpha_i = 2$, so that if 3 is a prime factor of s

$$\begin{aligned}\prod_{i=1}^{\ell}\frac{1}{(q_i^{2\alpha_i+1} - 1)^{\frac{1}{2}}}\frac{1}{(q_i-1)^{\frac{1}{2}}} &= \prod_{i=1}^{\ell}\frac{1}{(q_i^{2\alpha_i+1}-1)}[1 + q_i + q_i^2 + ... + q_i^{2\alpha_i}]^{\frac{1}{2}} \\ &= \left(\frac{11}{242}\right)^{\delta_{q_i,3}\delta_{\alpha_i,2}} \times \prod_{\substack{i=1\\(q_i,2\alpha_i+1)\neq(3,5)}}^{\ell}\frac{1}{(q_i^{2\alpha_i+1}-1)}[1 + q_i + q_i^2 + ... + q_i^{2\alpha_i}]^{\frac{1}{2}}\end{aligned} \tag{4}$$

From equation (3), the non-existence of odd perfect numbers can be deduced if

$$\begin{aligned}2(4k+1)\prod_i(1 + q_i + q_i^2 + ...\ q_i^{2\alpha_i}) \\ \cdot\ (1 + (4k+1) + (4k+1)^2 + ...\ (4k+1)^{(4m+1)}) \\ = 2(4k+1)\prod_i\frac{q_i^{2\alpha_i+1} - 1}{q_i - 1}\frac{(4k+1)^{4m+2} - 1}{4k}\end{aligned} \tag{5}$$

is not the square of an integer, with $q_i \neq 3$ or $\alpha_i \neq 2$.

Since the repunit $\frac{x^n - 1}{x - 1}$ is the Lucas sequence

$$U_n(a,b) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{6}$$

with $\alpha = x$, $\beta = 1$, derived from the second-order recurrence relation

$$U_{n+2}(a,b) \;=\; a\, U_{n+1}(a,b) \;-\; b\, U_n(a,b) \tag{7}$$

where $a = \alpha + \beta = x + 1$ and $b = \alpha\beta = x$, the rationality condition can be applied equally well to the product $\left[2(4k+1)\prod_{i=1}^{\ell}\ U_{2\alpha_i+1}(q_i+1, q_i)\cdot U_{4m+2}(4k+2, 4k+1)\right]^{\frac{1}{2}}$.

The number of square-full integers up to N is $N^{\frac{1}{2}} - \frac{3}{2}N^{-1} + O(N^{-\frac{3}{2}})$. With a lower bound of 10^{300} for an odd perfect number [10], it follows that $2(4k+1)\cdot\prod_{i=1}^{\ell}\ (q_i^{2\alpha_i} + O(q_i^{2\alpha_i-1}))\cdot((4k+1)^{4m+1} + O((4k+1)^{4m})) > 10^{301}$. Given a lower bound of 10^6 for the largest prime factor [11], 10^4 for the second largest prime factor and 10^2 for the third largest prime factor of N [12], the density of prime products $(4k+1)\prod_{i=1}^{\ell}\ q_i$, given by $\prod_{i=1}^{\ell}\ \frac{1}{ln\ q_i} \times \frac{1}{ln\ (4k+1)}$, is bounded above by 8.032×10^{-5} when there are eight different prime factors [2] and 1.004×10^{-6} when there are eleven different prime factors not including 3 [13][14]. Given that the probability of an integer being a square is independent of it being expressible in terms of a product of repunits, the density of square-full numbers having the form $2(4k+1)\ \sigma\left((4k+1)^{4m+1}\prod_i\ q_i^{2\alpha_i}\right)$ in the interval $[N^*, N^*+N_0]$, where $N^* > 10^{301}$ and N_0 is a fixed number, is bounded above by 3.28×10^{-159} when there are at least eight different prime factors and 5.13×10^{-163} when N is relatively prime to 3 and has more than ten different prime factors.

Although neither the numerator or the denominator will be squares of integers when $q_i \neq 3$ or $\alpha_i \neq 2$, there still remains the possibility that the terms could be equal multiples of different squares. Since the repunit $\frac{x^n\ -\ 1}{x\ -\ 1}$ is the Lucas sequence

$$U_n(a,b) \;=\; \frac{\alpha^n\ -\ \beta^n}{\alpha\ -\ \beta} \tag{8}$$

with $\alpha = x$, $\beta = 1$, derived from the second-order recurrence

$$U_{n+2}(a,b) \;=\; a\, U_{n+1}(a,b) \;-\; b\, U_n(a,b) \tag{9}$$

where $a = \alpha + \beta = x + 1$ and $b = \alpha\beta = x$, the rationality condition can be applied to the $\left[2(4k+1)\prod_{i=1}^{\ell}\ U_{2\alpha_i+1}(q_i+1, q_i)\cdot U_{4m+2}(4k+2, 4k+1)\right]^{\frac{1}{2}}$.

3. Proof of the non-existence of odd perfect numbers for a special class of integers

The even repunit $\frac{(4k+1)^{4m+2}-1}{4k}$ contains only a single power of 2 since $1+(4k+1)+(4k+1)^2+...+(4k+1)^{4m+1} \equiv 4m+2 \equiv 2 \ (mod\ 4)$. Thus, the rationality condition can be applied to a product of odd numbers $\left[(4k+1)\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1, q_i)\frac{1}{2}U_{4m+2}(4k+2, 4k+1)\right]^{1/2}$. Suppose

$$\begin{aligned} \prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \cdot \left[\frac{8k(4k+1)}{(4k+1)^{4m+2}-1}\right] &= \frac{r^2}{t^2} \\ \prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1}\ (4k+1)\ t^2 &= \frac{(4k+1)^{4m+2}-1}{8k}\ r^2 \end{aligned} \tag{10}$$

with $gcd(r,t) = 1$. If $gcd\left(\frac{(4k+1)^{4m+2}-1}{8k}, \frac{q_i^{2\alpha_i+1}-1}{q_i-1}\right) = 1$ for all i, the relation (10) requires $\left.\frac{(4k+1)^{4m+2}-1}{8k}\right| t^2$ or equivalently $\frac{(4k+1)^{4m+2}-1}{8k} = \sigma_\ell \tau_\ell^2$ where $\sigma_\ell \tau_\ell | t$. The substitution $t = \sigma_\ell \tau_\ell u$ gives

$$\begin{aligned} (4k+1)\ \prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}}{q_i-1} \cdot (\sigma_\ell \tau_\ell u)^2 &= \sigma_\ell \tau_\ell^2 \cdot r^2 \\ (4k+1)\ \prod_{i=1}^{\ell} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \sigma_l u^2 &= r^2 \end{aligned} \tag{11}$$

which, in turn, requires that $(4k+1)\ \prod_{i=1}^{\ell}\ \frac{q_i^{2\alpha_i+1}-1}{q_i-1} = \sigma_l\ v^2$ and $r = \sigma_l v u$, so that $\sigma_l u | r$ and $\sigma_l u | t$, contrary to the original assumption that r and t are relatively prime unless $\sigma_\ell = u = 1$. The rationality condition reduces to the existence of solutions to the equation

$$\frac{x^n-1}{x-1} = 2y^2 \qquad x \equiv 1\ (mod\ 4),\ n \equiv 2\ (mod\ 4) \tag{12}$$

This relation is equivalent to the two conditions $\frac{x^{2m+1}-1}{x-1} = y_1^2$, $\frac{x^{2m+1}+1}{2} = y_2^2$, $y = y_1 y_2$, $(y_1, y_2) = 1$ since $gcd(x^{2m+1}-1, x^{2m+1}+1) = 2$. It can be verified that there are no integer solutions to these simultaneous Diophantine equations, implying that when $\frac{(4k+1)^{4m+2}-1}{8k}$ satisfies the gcd condition given above, the square root of $\left[(4k+1)\prod_{i=1}^{\ell} U_{2\alpha_i+1}(q_i+1, q_i)\frac{1}{2}U_{4m+2}(4k+2, 4k+1)\right]$ is not a rational number and there is no odd perfect number of the form with this constraint on the pair $(4k+1, 4m+2)$.

4. Prime power divisors of Lucas sequences and Catalan's conjecture

The number of distinct prime divisors of $\frac{q^n-1}{q-1}$ is bounded below by $\tau(n) - 1$ if $q > 2$, where $\tau(n)$ is the number of natural divisors of n [1][15]. The characteristics of these prime divisors can be deduced from the properties of Lucas sequences. Since the repunits $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ have only odd prime divisors, the proofs in the following sections will have general validity, circumventing any exceptions corresponding to prime $q = 2$.

For a primary recurrence relation, defined by the initial values $U_0 = 0$ and $U_1 = 1$, denoting the least positive integer n such that $U_n(a, b) \equiv 0 \ (mod\, p)$, the rank of apparition, by $\alpha(a, b, p)$, it is known that $\alpha(x + 1, x, p) = ord_p(x)$ [16].

The extent to which the arguments a and b determine the divisibility of $U_n(a, b)$ [17][18] can be summarized as follows:

Let p be an odd prime.
If $p|a$, $p|b$, then $p|U_n(a, b)$ for all $n > 1$.
If $p \nmid a$, $p|b$, then either $p|U_n(a, b),\ n \geq 1$ or $p \nmid U_n(a, b)$ for any $n \geq 1$.
If $p|a$ and $p \nmid b$, then $p|U_n(a, b)$ for all even n or all odd n or $p \nmid U_n(a, b)$ for any $n \geq 1$.
If $p \nmid a$, $p \nmid b$, $p|D = a^2 - 4b$, then $p|U_n(a, b)$ when $p|n$.
If $p \nmid abD$, then $p|U_{p-\left(\frac{D}{p}\right)}(a, b)$.

For the Lucas sequence $U_n(q+1, q)$, there is no prime which divides both q and $q+1$, and since only q is a divisor of the second parameter, there are no prime divisors of $U_n(q+1, q)$ from this category because $\frac{q^n-1}{q-1} \equiv 1 \ (mod\, q)$. If $p|(q+1)$, then $\frac{q^n-1}{q-1} \equiv \frac{1-(-1)^n}{2} \equiv 0 \ (mod\, p)$ when n is even. However, $p \nmid \frac{q^n-1}{q-1}$ with n odd, and therefore, prime divisors from this class are not relevant for the study of the product of repunits with odd exponents.

When $a = q + 1$ and $b = q$, $D = (q - 1)^2$ and if $p|(q - 1)$, then $p|\ U_n(q + 1, q)$ when $p|n$. However, $p^2 \nmid \frac{q^p-1}{q-1}$, and under this condition, $p^2 \nmid \frac{q^n-1}{q-1}$ unless $n = cp^2$. More generally, denoting the power of p which exactly divides a by $p^{v_p(a)}$, it can be deduced that $v_p\left(\frac{q^n-1}{q-1}\right) = v_p(n)$ if $p|(q - 1)$ and $\alpha(q + 1, q, p) = p$ [9][15].

From the last property, it follows that $\alpha(a, b, p)| \left(p - \left(\frac{D}{p}\right)\right)$ When $p \nmid (q-1)$, $\left(\frac{D}{p}\right) = 1$ and $\alpha(q + 1, q, p)|(p - 1)$. If $p^2 \nmid \frac{q^{p-1}-1}{q-1}$, then $\alpha(q + 1, q, p^2) = p\ \alpha(q + 1, q, p)$ so that $\alpha(q + 1, q, p^2)|p(p - 1)$. If $p^2 \Big| \frac{q^{p-1}-1}{q-1}$, $\alpha(q + 1, q, p^2) = \alpha(q + 1, q, p)|p - 1$ [19][20]. Thus a

repunit with primitive divisor p is also divisible by p^2 if $Q_q \equiv 0 \ (mod\, p)$ where $Q_a = \frac{a^{p-1}-1}{p}$ is the Fermat quotient.

Since $q^n - 1 = \prod_{d|n} \Phi_d(q)$ where $\Phi_n(q)$ is the n^{th} cyclotomic polynomial, it can be shown that the largest arithmetic primitive factor [21]-[23] of $q^n - 1$ when $q \geq 2$ and $n \geq 3$ is

$$\begin{array}{ll} \Phi_n(q) & if\ \Phi_n(q)\ and\ n\ are\ relatively\ prime \\ \dfrac{\Phi_n(q)}{p} & if\ a\ common\ prime\ factor\ p\ of\ \Phi_n(q)\ and\ n\ exists \end{array} \tag{13}$$

In the latter case, if $n = p^f p'^{f'} p''^{f''}$ ... is the prime factorization of n, then $\Phi_n(q)$ is divisible by p if and only if $e = \frac{n}{p^f} = ord_p(q)$ when $p \nmid (q-1)$, and moreover, $p \parallel \Phi_{ep^f}(q)$ when $f > 0$ [1].

Division by $q-1$ does not alter the arithmetic primitive factor, since it is the product of the primitive divisors of $q^n - 1$, which are also the primitive divisors of $\frac{q^n-1}{q-1}$. For all primitive divisors, $p' \nmid (q-1)$, so that $(p')^h \Big| \frac{q^n-1}{q-1}$ if $(p')^h | q^n - 1$ and the arithmetic primitive factor again would include $(p')^h$. The imprimitive divisors would be similarly unaffected because the form of the index $n = ep^f$ prevents $q-1$ from being a divisor of $\Phi_n(q)$ when $p \nmid (q-1)$. If $p|(q-1)$, the rank of apparition for the Lucas sequence $\{U_n(q+1, q)\}$ is p, so that it is consistent to set $n = p^{f+1}$. Then, $p \parallel \Phi_{p^{f+1}}(q)$ and the arithmetic primitive factor is $\frac{\Phi_{p^{f+1}}(q)}{p}$.

If $(q_i - 1) \nmid \Phi_{n_i}(q_i)$, the product of the arithmetic primitive factors of each repunit $\frac{q_i^{n_i} - 1}{q_i - 1}$ and $\frac{(4k+1)^{4m+2}-1}{4k}$ in the expression (5) is

$$\frac{\Phi_{n_1}(q_1)}{p_1} \ \frac{\Phi_{n_2}(q_2)}{p_2} \ ... \ \frac{\Phi_{n_\ell}(q_\ell)}{p_\ell} \ \times \ \left[\frac{\Phi_{4m+2}(4k+1)}{p_{\ell+1}}\right] \tag{14}$$

where the indices are odd numbers $n_i = 2\alpha_i + 1$, p_i, $i = 1, ..., l$, represents the common factor of n_i and $\Phi_{n_i}(q_i)$, and $p_{\ell+1}$ is a common factor of $4m+2$ and $\Phi_{4m+2}(4k+1)$. Division of $\Phi_{n_i}(q_i)$ by the prime p_i is necessary only when $gcd(n_i, \Phi_{n_i}(q_i)) \neq 1$, and $p_i = P\left(\frac{n_i}{gcd(3,n_i)}\right)$, where $P(n)$ represents the largest prime factor of n [17] [24]-[27].

Theorem 1. The arithmetic primitive factors of the repunits with different prime bases could be equal only if the exponents are different, with possible exceptions being determined by the solutions to the equation $\frac{q_j^n - 1}{q_i^n - 1} = p$, $q_i \neq q_j$ with q_i, q_j and p prime.

Proof. Consider the following four cases:

I. The arithmetic primitive factors of $q_i^{n_i}-1$ and $q_j^{n_j}-1$ are $\Phi_{n_i}(q_i)$ and $\Phi_{n_j}(q_j)$.

Since $\Phi_n(x)$ is a strictly increasing function for $x \geq 1$ [28], $\Phi_n(q_j) > \Phi_n(q_i)$ when q_j is the larger prime, and equality of $\Phi_{n_i}(q_i)$ and $\Phi_{n_j}(q_j)$ could only be achieved, if at all feasible, when $n_i \neq n_j$.

II. The arithmetic primitive factors of $q_i^{n_i}-1$ and $q_j^{n_j}-1$ are $\Phi_{n_i}(q_i)$ and $\frac{\Phi_{n_j}(q_j)}{p_j}$.

Comparing $\Phi_n(q_i)$ and $\frac{\Phi_n(q_j)}{p}$, $p = p_j$ is a common factor of n and $\Phi_n(q_j)$ but it does not divide $\Phi_n(q_i)$. It follows that the relation $\Phi_n(q_i) = \frac{\Phi_n(q_j)}{p}$ could only hold if $p \parallel \Phi_n(q_j)$. The prime decomposition of e as $\rho_1^{r_1}...\rho_s^{r_s},\ gcd(\rho_t, p) = 1,\ t = 1, ..., s$, leads to the following expressions for $\Phi_n(q_i)$ and $\Phi_n(q_j)$,

$$\Phi_n(q_i) = \Phi_{ep^f}(q_i) = \frac{\Phi_e(q_i^{p^f})}{\Phi_e(q_i^{p^{f-1}})}$$

$$= \frac{\prod_{\substack{k\ even \\ k\geq 0}} \prod_{\substack{t_k>...>t_1 \\ t_k \leq s}} \left[q_i^{\frac{ep^f}{\rho_{t_1}\cdots\rho_{t_k}}} - 1 \right]}{\prod_{\substack{\tilde{k}\ odd \\ \tilde{k}\geq 1}} \prod_{\substack{t_{\tilde{k}}>...>t_1 \\ t_{\tilde{k}} \leq s}} \left[q_i^{\frac{ep^f}{\rho_{t_1}\cdots\rho_{t_{\tilde{k}}}}} - 1 \right]} \cdot \frac{\prod_{\substack{\tilde{k}\ odd \\ \tilde{k}\geq 1}} \prod_{\substack{t_{\tilde{k}}>...>t_1 \\ t_{\tilde{k}} \leq s}} \left[q_i^{\frac{ep^{f-1}}{\rho_{t_1}\cdots\rho_{t_{\tilde{k}}}}} - 1 \right]}{\prod_{\substack{k\ even \\ k\geq 1}} \prod_{\substack{t_k>...>t_1 \\ t_k \leq s}} \left[q_i^{\frac{ep^{f-1}}{\rho_{t_1}\cdots\rho_{t_k}}} - 1 \right]}$$

$$\Phi_n(q_j) = \Phi_{ep^f}(q_j) = \frac{\Phi_e(q_j^{p^f})}{\Phi_e(q_j^{p^{f-1}})}$$

$$= \frac{\prod_{\substack{k\ even \\ k\geq 0}} \prod_{\substack{t_k>...>t_1 \\ t_k \leq s}} \left[q_j^{\frac{ep^f}{\rho_{t_1}\cdots\rho_{t_k}}} - 1 \right]}{\prod_{\substack{\tilde{k}\ odd \\ \tilde{k}\geq 1}} \prod_{\substack{t_{\tilde{k}}>...>t_1 \\ t_{\tilde{k}} \leq s}} \left[q_j^{\frac{ep^f}{\rho_{t_1}\cdots\rho_{t_{\tilde{k}}}}} - 1 \right]} \cdot \frac{\prod_{\substack{\tilde{k}\ odd \\ \tilde{k}\geq 1}} \prod_{\substack{t_{\tilde{k}}>...>t_1 \\ t_{\tilde{k}} \leq s}} \left[q_j^{\frac{ep^{f-1}}{\rho_{t_1}\cdots\rho_{t_{\tilde{k}}}}} - 1 \right]}{\prod_{\substack{k\ even \\ k\geq 1}} \prod_{\substack{t_k>...>t_1 \\ t_k \leq s}} \left[q_j^{\frac{ep^{f-1}}{\rho_{t_1}\cdots\rho_{t_k}}} - 1 \right]} \qquad (15)$$

Since $e = ord_p(q_j)$, it follows that $p|(q_j^e - 1)$, and if $q_j^e = 1 + pk_j\ (mod\ p)$, then $(q_j^e)^{p^f} = (1+pk_j)^{p^f} \equiv 1 + p^f pk_j \equiv 1\ (mod\, p^{f+1})$. Thus, $p^{f+1}|(q_j^{ep^f} - 1)$ and $p^f|(q_j^{ep^{f-1}} - 1)$, while $p \not| \left(q_j^{\frac{e}{\rho_{t_1}\cdots\rho_{t_k}}p^f} - 1 \right)$. Let $H(f) \geq f+1$ denote the exponent such that

$p^{H(f)} \parallel (q_j^{ep^f} - 1)$. Since $q_j^{ep^{f-1}} \equiv 1 \ (mod\ p^{H(f-1)})$, $q_j^{ep^f} = (1 + p^{H(f-1)} k'_j)^p \equiv 1 + p \cdot p^{H(f-1)} k'_j \equiv 1 \ (mod\ p^{H(f-1)+1})$. Consequently, $H(f) - H(f-1) = 1$, which is consistent with $\Phi_n(q_j)$ being exactly divisible by p.

Although $\Phi_n(q_i)$ and $\frac{\Phi_n(q_j)}{p}$ are not divisible by p, consider a primitive prime factor p' of $\Phi_n(q_i)$. It must divide some factor $q_i^{\overline{\frac{n}{\rho_{t_1} \cdots \rho_{t_k}}}} - 1$ in the expression for $\Phi_n(q_i)$, and thus, it will also divide $q_i^{\overline{\frac{n}{\rho_{t_1} \cdots \rho_{t_\ell}}}} - 1,\ \ell < k$. Since the exponent of $q_i^{\overline{\frac{n}{\rho_{t_1} \cdots \rho_{t_\ell}}}} - 1$ in $\Phi_n(q_i)$ is $(-1)^l$, there will be 2^{k-1} factors in the numerator and 2^{k-1} factors in the denominator divisible by p'. When $k \geq 1$, the factors of p' exactly cancel because each term $q_i^{\overline{\frac{n}{\rho_{t_1} \cdots \rho_{t_\ell}}}} - 1$ is divisible by the same power of p'. The exception occurs when $p' | q_i^n - 1$ only; if $p_a'^{f_a} \parallel q_i^n - 1$, then $p_a'^{f_a} \parallel \Phi_n(q_i)$ [29]. Equivalence of $\Phi_n(q_i)$ and $\frac{\Phi_n(q_j)}{p}$ requires that the prime power divisors of these quantities are equal, so that $p_a'^{f_a} \left\| \frac{\Phi_n(q_j)}{p} \right.$ for all primes $\{p'_a\}$. However, if $p_a'^{f_a} \parallel q_j^n - 1$, then $q_i^n - 1$ and $q_j^n - 1$ have the same primitive prime power divisors. The imprimitive prime divisor p which divides $q_j^n - 1$ might also divide $q_i^n - 1$, although overall cancellation of p in $\Phi_n(q_i)$ requires that $p^r \left| q_i^{\overline{\frac{n}{\rho_{t_1} \cdots \rho_{t_k}}}} - 1 \right.$ for some $k \geq 1$ and $p^{r-1} \left| q_i^{\overline{\frac{n}{p\rho_{t_1} \cdots \rho_{t_k}}}} - 1 \right.$. When $p^r \parallel q_i^n - 1$ and $\frac{q_j^n - 1}{q_i^n - 1} = p^{H(f)-r}$

$$q_i^n - 1 = \kappa u_1^{H(f)-r} \qquad q_j^n - 1 = \kappa u_2^{H(f)-r} \qquad \frac{u_2}{u_1} = p \tag{16}$$

Integer solutions of $w = y^m,\ y \geq 2,\ m \geq 2$ can be written as $w = x^n,\ x \geq 2$ with $m|n$. Since $y|x^n$, $y \nmid (x^n - 1)$ because $y \geq 2$. The nearest integers to x^n having a similar form, $\{(x-1)^n, (x+1)^n, (x+1)^{n-1}, (x-1)^{n+1}\}$ do not provide a counterexample to the conclusion since none of them are divisible by y. Furthermore, $x^n - (x-1)^n > 1$, $(x+1)^n - x^n > 1$, $|(x+1)^{n-1} - x^n| > 1,\ x \geq 2, n \geq 4; x \geq 3, n \geq 3$ and $|x^n - (x-1)^{n+1}| > 1,\ x \geq 2, n \geq 3$ so that none of these integers will have the form $y^m \pm 1$. The exception occuring when $x = y = 2,\ m = n = 3$ is the statement of Catalan's conjecture, that $(X, Y, U, V) = (3, 2, 2, 3)$ is the only integer solution of $X^U - Y^V = 1$. Thus, if $\kappa = 1$, any non-trivial solution to equation (16) is constrained by the condition $H(f) - r = 1$, which implies that $\frac{q_j^n - 1}{q_i^n - 1} = p$. Since the odd primes q_i, q_j and the exponent n in the prime decomposition of N must be greater than or equal to 3, this restriction is consistent with Catalan's conjecture.

When $\kappa \neq 1$, it may be noted that for $q_i,\ q_j \gg 1$, $\frac{q_j^n - 1}{q_i^n - 1} \simeq \left(\frac{q_j}{q_i}\right)^n \neq p^h$. Exceptional solutions to equation (16) occur, for example, when $h = 1$; they include $\{(q_i, q_j; n; p) = (3, 5; 2; 3),\ (5, 7; 2; 2),\ (5, 11; 2; 5),\ (5, 13; 2; 7),\ (11, 19; 2; 3),\ (7, 23; 2; 11),\ (11, 29; 2; 7),$

$(29, 41; 2; 2)\}$. Since $q_i \neq q_j$, with the exception of the non-trivial solutions to equation (16), it would be necessary to set $n_i \neq n_j$ to obtain equality between $\Phi_{n_i}(q_i)$ and $\frac{\Phi_{n_j}(q_j)}{p}$.

III. The arithmetic primitive factors of $q_i^{n_i} - 1$ and $q_j^{n_j} - 1$ are $\frac{\Phi_{n_i}(q_i)}{p_i}$ and $\Phi_{n_j}(q_j)$.

The proof of the necessity of $n_i \neq n_j$ for any equality between the arithmetic primitive factors is similar to that given in Case II with the roles of i and j interchanged.

IV. The arithmetic primitive factors of $q_i^{n_i} - 1$ and $q_j^{n_j} - 1$ are $\frac{\Phi_{n_i}(q_i)}{p_i}$ and $\frac{\Phi_{n_j}(q_j)}{p_j}$.

Since $p_i = gcd(n_i, \Phi_{n_i}(q_i))$ and $p_j = gcd(n_j, \Phi_{n_j}(q_j))$, $\Phi_{n_i}(q_i)$ and $\Phi_{n_j}(q_j)$ share a common factor if $n_i = n_j$. Thus, the primes p_i and p_j must be equal, and a comparison can be made between $\frac{\Phi_n(q_i)}{p}$ and $\frac{\Phi_n(q_j)}{p}$. Again, by the monotonicity of $\Phi_n(x)$, it follows that these quantities are not equal when q_i and q_j are different primes. Equality of the arithmetic prime factors could only occur if $n_i \neq n_j$. $\blacksquare$

5. The exponent of prime divisors of repunit factors in the rationality condition

Since all primitive divisors of $U_n(a, b)$ have the form $p = nk + 1$, it follows that $p \left| \frac{q^{\frac{(p-1)}{\iota(p)}} - 1}{q - 1} \right.$. If $\iota(p)$ is odd, where $\iota(p)$ is the residue index, the exponent $\frac{p-1}{\iota(p)}$ will be even for all odd primes p, whereas if $\iota(p)$ is even, the exponent $\frac{p-1}{\iota(p)}$ may be even or odd. Given that $p|U_{2\alpha_i+1}(q_i + 1, q_i)$, $\iota(p)$ is even and $p \left| \frac{q_i^{\frac{(p-1)}{2}} - 1}{q_i - 1} \right.$ implying $q_i^{\frac{p-1}{2}} \equiv 1\ (mod\ p)$ and $\left(\frac{q_i}{p}\right) = 1$. Moreover, if $\left(\frac{q_i}{p}\right) = \left(\frac{q_j}{p}\right) = 1$, $\left(\frac{q_i q_j}{p}\right) = 1$ implying that $p|(q_i q_j)^{\frac{p-1}{2}} - 1$. Thus, the Fermat quotient is $Q_{q_i q_j} = \frac{(q_i q_j)^{\frac{p-1}{2}} - 1}{p}\left((q_i q_j)^{\frac{p-1}{2}} + 1\right) = \mathcal{N}_{q_i q_j}(\mathcal{N}_{q_i q_j} p + 2)$ where $\mathcal{N}_q$ can be defined to be $\frac{q^{\frac{(p-1)}{2}} - 1}{p}$. By the logarithmic rule for Fermat quotients, $Q_{qq'} \equiv Q_q + Q_{q'}\ (mod\ p)$ [30], so that $\mathcal{N}_{q_i q_j} \equiv \mathcal{N}_{q_i} + \mathcal{N}_{q_j}\ (mod\ p)$.

Recalling that $\alpha(q_i + 1, q_i, p^2) \neq \alpha(q_i + 1, q_i, p)$ only when $p^2 \not\left| \frac{q_i^{p-1} - 1}{q_i - 1} \right.$, it is sufficient to prove that the Fermat quotient $Q_{q_i} \neq 0\ (mod\ p)$ to show that the p^2 is not a divisor of the repunit $\frac{q_i^{2\alpha_i+1} - 1}{q_i - 1}$. It has been established that $q^{p-1} - 1 \equiv p\left(\mu_1 + \frac{\mu_2}{2} + ... + \frac{\mu_{p-1}}{p-1}\right)$ $(mod\ p^2)$, where $\mu_i \equiv \left[\frac{-i}{p}\right]\ (mod\ q)$ is the minimum positive integer congruent to $(-i \cdot p^{-1})\ (mod\ q)$ [31][32]. Since $\mu_i \neq 0$ in general, except when $i = q$, it follows that $q^{p-1} - 1 \neq 0\ (mod\ p^2)$ except for $p - 1$ values of q between 1 and $p^2 - 1$.

By Hensel's lemma [33][34], each of the integers between 1 and $p-1$, which satisfy $x^{p-1}-1 \equiv 0 \ (mod\ p)$ generate the $p-1$ solutions to the congruence equation

$$(x')^{p-1}-1 \equiv 0 \ (mod\ p^2) \tag{17}$$

through the formula

$$x' \ = \ x + \left(\frac{-g_1(x)p}{(p-1)q^{p-2}} \right) \ (mod\ p^2) \tag{18}$$

with $x^{p-1}-1 \equiv g_1(x)p \ (mod\ p^2)$.

Since $\varphi(p^2) = p(p-1)$, a set of $p-1$ solutions to equation (17) can also be labelled as $c^p \ (mod\ p^2), \ 1 \le c \le p-1$, since $(c^p)^{p-1} = c^{p(p-1)} = c^{\varphi(p^2)} \equiv 1 \ (mod\ p^2)$. Each power c^p is different, because $c_1^p \equiv c_2^p \ (mod\ p^2)$ implies $c_1 = c_2$ since $p^2 \nmid (c_3^p - 1)$ for any c_3 between 1 and $p-1$.

Theorems concerning the Fermat quotient $\frac{q^r-1}{q-1}$ can be extended to quotients of the type $\frac{q^{nr}-1}{q^n-1}$. It has been proven, for example, that $p \Big\| \frac{q^{nr}-1}{q^n-1}, \ p \nmid r, \ p \nmid q^n-1$, then $Q_q = \frac{q^{p-1}-1}{p} \not\equiv 0 \ (mod\ p)$ [35], and more generally, if $p^h \ \Big\| \ \frac{q^{nr}-1}{q^n-1}, \ p \nmid r, \ p \nmid q^n-1$, then $q^{p-1} \nmid 1 (mod\ p^{h+1})$.

When $p|(q^r-1)$, the following lemma is obtained.

Lemma. For any prime p which is a primitive divisor of $U_{2\alpha_i+1}(q_i+1, q_i)$, $p \nmid \frac{q_i^{p-1}-1}{q_i^{(2\alpha_i+1)}-1}$, and if $p^h \parallel U_{2\alpha_i+1}(q_i+1, q_i)$, then $p^h \Big\| \frac{q_i^{2\alpha_i+1}-1}{q_i^{\frac{(2\alpha_i+1)}{s}}-1}$ for any non-trivial divisor s of $2\alpha_i+1$.

Proof. Defining the residue index $\iota_i(p)$ by $p-1 = (2\alpha_i+1)\iota_i(p)$, then

$$p \ \Bigg| \ \frac{q_i^{p-1}-1}{q_i-1} = \left[\frac{q^{(2\alpha_i+1)\iota_i(p)}-1}{q_i^{(2\alpha_i+1)}-1} \right] \cdot \left[\frac{q^{(2\alpha_i+1)}-1}{q_i-1} \right] \tag{19}$$

Suppose that $p | \frac{q_i^{(2\alpha_i+1)\iota_i(p)}-1}{q_i^{2\alpha_i+1}-1}$. Then, by equation (19), $p^2 \Big| \frac{q_i^{p-1}-1}{q-1}$. By a theorem on congruences, if $q^e \equiv 1 \ (mod\ p)$, where $e|(p-1)$ and $q^{p-1} \equiv 1 \ (mod\ p^2)$, then $q^e \equiv 1 \ (mod\ p^2)$ [28], so that $p^2 \Big| \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$. Consequently, $p^3 \Big| \frac{q_i^{p-1}-1}{q_i-1}$. The theorem on congruences can be extended to larger prime powers: $q^e \equiv 1 \ (mod\ p^n)$ and $q^{p-1} \equiv 1 \ (mod\ p^{n+1})$, then

$q^e \equiv 1 \ (mod\ p^{n+1})$. From the first congruence relation, $q^e = 1 + k'p^n$ for some integer k'. Raising this quantity to the power $\frac{p-1}{e}$, it follows that

$$1 \ \equiv \ q^{p-1} = (q^e)^{\frac{p-1}{e}} = (1 + k'p^n)^{\frac{p-1}{e}} \equiv 1 + k'p^n \frac{p-1}{e} \ (mod\ p^{n+1}) \qquad (20)$$

Since $\frac{p-1}{e} < p$, the integer k' must be a multiple of p. Thus, $q^e = 1 + k''p^{n+1} \equiv 1 \ (mod\ p^{n+1})$. By the generalized congruence theorem, $p^3 \Big| \frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ and equation (20) in turn implies that $p^4 \Big| \frac{q_i^{p-1}-1}{q_i-1}$. Since this process can be continued indefinitely to arbitrarily high powers of the prime p, a contradiction is obtained once the maximum exponent is greater than h, where $p^h \Big| \frac{q_i^{p-1}-1}{q_i-1}$. Therefore, $p \not\Big| \frac{q_i^{(2\alpha_i+1)\iota_i(p)}-1}{q_i^{2\alpha_i+1}-1}$.

Similarly,

$$\frac{q_i^{2\alpha_i+1} - 1}{q_i - 1} = \left[\frac{q_i^{2\alpha_i+1} - 1}{q_i^{\frac{2\alpha_i+1}{s}} - 1}\right] \cdot \left[\frac{q_i^{\frac{2\alpha_i+1}{s}} - 1}{q_i - 1}\right] \qquad (21)$$

If s is a non-trivial divisor of $2\alpha_i+1$, then $p \not\Big| \frac{q_i^{\frac{2\alpha_i+1}{s}}-1}{q_i-1}$, because it is a primitive divisor of $U_{2\alpha_i+1}(q_i+1, q_i)$. Given that $p^h \parallel U_{2\alpha_i+1}(q_i+1, q_i)$, by equation (21), $p^h \Big\| \frac{q_i^{(2\alpha_i+1)}-1}{q_i^{\frac{(2\alpha_i+1)}{s}}-1}$. $\blacksquare$

Imprimitive prime divisors of $U_n(a, b)$ are characterized by the property that $p|U_d(a, b)$ for some $d|n$. The exponent of the imprimitive prime power divisor exactly dividing $\frac{q^n-1}{q-1}$ can be determined by a further lemma: if $p^h \Big| \frac{q^n-1}{q-1}$, then either $gcd(n, p-1) = 1$, $q \equiv 1 \ (mod\ p)$, $p^h|n \ (mod\ p)$ or $e = gcd(n, p-1) > 1$, $p^k|\Phi_e(q)$, $p^{h-k} \parallel n$ [15]. Since $v_p(\Phi_e(q)) = v_p(q^e - 1)$ if $p \nmid q-1$, the general formula [12] for the exponent of a prime divisor of a repunit is

$$v_p\left(\frac{q^n - 1}{q - 1}\right) \ = \begin{cases} v_p(q^e-1)+v_p(n) & e=ord_p(q)|n, \ e>1 \\ v_p(n) & p|q-1 \\ 0 & otherwise \end{cases} \qquad (22)$$

The exponent also can be deduced from the congruence properties of q-numbers $[n] = \frac{q^n-1}{q-1}$

and q-binomial coefficients [36], as it equals $s = \epsilon_0 h + \epsilon_1 + ... + \epsilon_{k-1}$ where $p^h \parallel q^e - 1$ and

$$
\begin{aligned}
n - 1 &= a_0 + e(a_1 + a_2 p + ... + a_k p^{k-1}) \\
n &= b_0 + e(b_1 + b_2 p + ... + b_k p^{k-1}) \\
a_0 \leq e - 1,\ a_i &\leq p - 1,\ \ i = 1, ..., k - 1 \\
b_0 \leq e - 1,\ b_i &\leq p - 1,\ \ i = 1, ..., k - 1 \\
a_0 + 1 &= \epsilon_0 e + b_0 \\
\epsilon_0 + a_1 &= \epsilon_1 p + b_1 \\
&... \\
\epsilon_{k-2} + a_{k-1} &= \epsilon_{k-1} p + b_{k-1} \\
\epsilon_{k-1} + a_k &= b_k
\end{aligned}
\tag{23}
$$

with ϵ_i equal to 0 or 1, which is consistent with equation (21) because $\epsilon_0 = 1$ and $v_p(n) = \epsilon_1 + ... + \epsilon_{k-1}$.

Specializing to the case of $h = 2$, it follows that if the quotient $\frac{q^n-1}{q-1}$ is exactly divisible by p^2, then

(i) $gcd(n, p-1) = 1$, $p|(q-1)$ or $p \nmid q^n - 1$, $p^2 \parallel n$
(ii) $p \parallel \Phi_e(q)$, where $e = \alpha(q+1, q, p)$ is the rank of apparition of p, $p \parallel n$
(iii) $p^2 \parallel \Phi_e(q)$, $p \nmid n$

and the only indices n_i which allow for exact divisibility of $\frac{q_i^{n_i}-1}{q_i-1}$ by p^2 are $n_i = \mu p^2$, when $p|(q_i - 1)$ or $e_i \nmid n_i$, $n_i = \mu e_i p$ when $p \parallel \Phi_{e_i}(q_i)$ and $n_i = \mu e_i$ when $p^2 \parallel \Phi_{e_i}(q_i)$. Since n_i is odd, the three categories can be defined by the conditions: (i) $n_i = \mu p^2$, (ii) $n_i = \mu e_i p$, p is a primitive divisor of $\frac{q_i^{e_i}-1}{q_i-1}$, $Q_{q_i} \not\equiv 0\ (mod\ p)$ (iii) p is a primitive divisor of $\frac{q_i^{e_i}-1}{q_i-1}$, $Q_{q_i} \equiv 0\ (mod\ p)$.

6. A proof by the method of induction of the non-existence of a generic set of primes satisfying the rationality condition

The equation

$$
a\frac{x^m - 1}{x - 1} = b\frac{y^n - 1}{y - 1} \tag{24}
$$

is known to have finitely many integer solutions for m, n, x, y, given a and b such that $gcd(a, b) = 1$, $a(y-1) \neq b(x-1)$, and $max(m, n, x, y) < C$ where C is an effectively

computable number depending on a, b and F where $|x - y| < F\frac{z}{(log\ z)^2(log\ log\ z)^3}$ with $z = max(x, y)$ [37][38]. Using this relation to re-express $\frac{q_i^{2\alpha_i+1}-1}{q_i-1}$ in terms of $\frac{(4k+1)^{4m+2}-1}{4k}$, it can be established that there are unmatched primes in the product of the repunits (5) and that the square root of this expression is irrational for several different categories of prime divisors $\{q_i,\ i = 1, ..., \ell;\ 4k + 1\}$.

Theorem 2. The square-root expressions $\sqrt{2(4k+1)}\left[\frac{q_1^{2\alpha_1+1}-1}{q_i-1}...\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}\right]^{\frac{1}{2}}$ $\cdot\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\frac{1}{2}}$ are not rational numbers for the following sets of primes $\{q_i, i = 1, ..., \ell;$ $4k + 1\}$ and exponents $2\alpha_i + 1$:

(i) For sets of primes with the number of elements given by consecutive integers, $\{q_i, i = 1, ...\ell - 1,\ 4k + 1\}$ and $\{q'_j,\ j = 1, ..., \ell,\ 4k' + 1\}$, there cannot be odd integers of the form $N_1 = (4k + 1)^{4m+1}q_1^{2\alpha_1}...q_{\ell-1}^{2\alpha_{\ell-1}}$ and $N_2 = (4k' + 1)^{4m'+1}(q'_1)^{2\alpha'_1}...(q'_\ell)^{2\alpha'_\ell}$ such that both $\frac{\sigma(N_1)}{N_1} = 2$ and $\frac{\sigma(N_2)}{N_2} = 2$.

(ii) Setting $\alpha_j = \alpha_\ell$, extra prime divisors p of the repunits $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, $j < \ell$ and $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$, where $p|\ (q_j - 1)$ but $p \nmid (q_\ell - 1)$, cannot be absorbed into the square factors if

$Q_{q_\ell} \not\equiv\ 0\ (mod\ p)$ or $p^{h'_\ell}\Big\|\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ with h'_ℓ odd. Similarly, if $p \nmid (q_j - 1)$ but $p|\ (q_\ell - 1)$, then an odd power of p divides the product of the two repunits if $Q_{q_j} \not\equiv 0\ (mod\ p)$ or $Q_{q_j} \equiv 0\ (mod\ p^{h'_j-1})$, $Q_{q_j} \not\equiv 0\ (mod\ p^{h'_j})$, with h'_j odd, and p remains an unmatched prime divisor.

(iii) When $n_j = 2\alpha_j + 1$ is set equal to $n_\ell = 2\alpha_\ell + 1$, the primitive prime divisors of $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ cannot not be matched to produce the square of a rational number if $\frac{q_\ell^{n_\ell}-1}{q_j^{n_\ell}-1} \neq \frac{y_2^2}{y_1^2}$, $y_1,\ y_2\ \in\ \mathbb{Z}$. This property is valid, for example, when $q_\ell^{\frac{n_\ell}{2}} <$ $gcd(q_j^{n_j} - 1, q_\ell^{n_\ell} - 1)$.

(iv) Additional prime divisors are introduced when the exponents are adjusted, so that, in general, there will be unmatched prime divisors in the products of the repunits $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$, $j < \ell$ and $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$ when $\alpha_j \neq \alpha_\ell$.

Proof.

Suppose $\{a_i\}$ and $\{b_i\}$ are defined by

$$\begin{aligned} a_1 \frac{q_1^{2\alpha_1+1}-1}{q_1-1} &= b_1 \frac{(4k+1)^{4m+2}-1}{4k} \\ a_2 \frac{q_2^{2\alpha_2+1}-1}{q_2-1} &= b_2 \frac{(4k+1)^{4m+2}-1}{4k} \\ &\vdots \\ a_\ell \frac{{q_\ell}^{2\alpha_\ell+1}-1}{q_\ell-1} &= b_\ell \frac{(4k+1)^{4m+2}-1}{4k} \end{aligned} \tag{25}$$

Then

$$\begin{aligned} &\sqrt{2(4k+1)} \left[\frac{q_1^{2\alpha_1+1}-1}{q_1-1} \frac{q_2^{2\alpha_2+1}-1}{q_2-1} \ldots \frac{{q_\ell}^{2\alpha_\ell+1}-1}{q_\ell-1} \frac{(4k+1)^{4m+2}-1}{4k} \right]^{\frac{1}{2}} \\ &\qquad = \sqrt{2(4k+1)}\, \frac{(b_1 b_2 \ldots. b_\ell)^{\frac{1}{2}}}{(a_1 a_2 \ldots a_\ell)^{\frac{1}{2}}} \cdot \left(\frac{(4k+1)^{4m+2}-1}{4k} \right)^{\frac{(\ell+1)}{2}} \end{aligned} \tag{26}$$

If

$$a_{ij} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} = b_{ij} \frac{q_j^{2\alpha_j+1}-1}{q_j-1} \tag{27}$$

define $\{a_{ij}\}$ and $\{b_{ij}\}$ with $gcd(a_{ij}, b_{ij}) = 1$,

$$\frac{b_1 b_2 b_3}{a_1 a_2 a_3} = \frac{b_{13}}{a_{13}} \frac{a_2}{b_2} \times \left(\frac{b_2 b_3}{a_2 a_3} \right)^2 \tag{28}$$

and

$$\frac{b_1 b_2 \ldots b_\ell}{a_1 a_2 \ldots a_\ell} = \frac{b_{1\ell}}{a_{1\ell}} \frac{a_2}{b_2} \ldots \frac{a_{\ell-1}}{b_{\ell-1}} \left(\frac{b_2 b_3 \ldots b_\ell}{a_2 a_3 \ldots a_\ell} \right)^2 \tag{29}$$

Since the fraction $\frac{b_1}{a_1}$ can be expressed in terms of $\frac{b_2}{a_2}$

$$\frac{b_1}{a_1} = \frac{b_2}{a_2} \frac{\rho_{12}^{r_{12}}}{\chi_{12}^{s_{12}}} = \frac{b_2}{a_2} \frac{\rho_{12}^{(r_{12})_0}}{\chi_{12}^{(s_{12})_0}} \frac{\rho_{12}^{(r_{12}-(r_{12})_0)}}{\chi_{12}^{(s_{12}-(s_{12})_0)}} \tag{30}$$

where $\rho_{12}^{r_{12}},\ \chi_{12}^{s_{12}}$ denote products of various powers of different primes, with $r_{12},\ s_{12}$ representing the sets of exponents, $(r_{12})_0,\ (s_{12})_0$ labelling a collection of exponents consisting of 0 or 1 and ρ_{12}, χ_{12} being products of these primes with all of the exponents equal to 1. The sets $(r_{12})_0,\ (s_{12})_0$ are chosen so that $r_{12} - (r_{12})_0 = 2\bar{r}_{12},\ s_{12} - (s_{12})_0 = 2\bar{s}_{12}$ represent even exponents. Since a similar relation exists between $\frac{a_2}{b_2}$ and $\frac{a_3}{b_3}$,

$$\frac{b_{13}}{a_{13}}\frac{a_2}{b_2} = \left[\frac{a_2}{b_2}\frac{\rho_{12}^{(r_{12})_0}}{\chi_{12}^{(s_{12})_0}}\frac{\rho_{23}^{(r_{23})_0}}{\chi_{23}^{(s_{23})_0}}\right]\left(\frac{\rho_{12}^{\bar{r}_{12}}}{\chi_{12}^{\bar{s}_{12}}}\right)^2\left(\frac{\rho_{23}^{\bar{r}_{23}}}{\chi_{23}^{\bar{s}_{23}}}\right)^2 \tag{31}$$

If $(r_{12})_0 = (s_{12})_0 = (r_{23})_0 = (s_{23})_0 = \{0\}$, then $\frac{b_{13}}{a_{13}}\frac{a_2}{b_2} \neq \frac{4k+1}{2}\cdot\square$ * because rationality of $\sqrt{2(4k+1)}\left[\frac{q_2^{2\alpha_2+1}-1}{q_2-1}\frac{(4k+1)^{4m+2}-1}{4k}\right]^{\frac{1}{2}}$ would imply that $\frac{\sigma(N)}{N} = 2$ where $N = (4k+1)^{4m+1}q_2^{2\alpha_2}$, contradicting the non-existence of odd perfect numbers with two prime factors.

If $(r_{12})_0 = (s_{12})_0 = \{0\}$, the expression in brackets is not $\frac{4k+1}{2}$ times the square of a rational number because

$$\frac{a_2}{b_2}\frac{\rho_{23}^{(r_{23})_0}}{\chi_{23}^{(s_{23})_0}} = \frac{a_2}{b_2}\frac{\rho_{23}^{r_{23}}}{\chi_{23}^{s_{23}}}\cdot\left(\frac{\rho_{23}^{2\bar{r}_{23}}}{\chi_{23}^{2\bar{s}_{23}}}\right)^{-1} = \frac{a_3}{b_3}\left(\frac{\chi_{23}^{\bar{s}_{23}}}{\rho_{23}^{\bar{r}_{23}}}\right)^2 \tag{32}$$

and $\frac{a_3}{b_3} \neq \frac{4k+1}{2}\cdot\square$, since $\sqrt{2(4k+1)}\left[\frac{q_3^{2\alpha_3+1}-1}{q_3-1}\frac{(4k+1)^{4m+2}-1}{4k}\right]^{\frac{1}{2}}$ is not rational and there is no odd integer N of the form $(4k+1)^{4m+1}q_3^{2\alpha_3}$ such that $\frac{\sigma(N)}{N} = 2$. A similar conclusion holds when $(r_{23})_0 = \{0\}$ and $(s_{23})_0 = \{0\}$.

If both fractions $\frac{\rho_{12}^{(r_{12})_0}}{\chi_{12}^{(s_{12})_0}}$, $\frac{\rho_{23}^{(s_{23})_0}}{\chi_{23}^{(s_{23})_0}}$ are non-trivial, at least one of the pair of exponents $((r_{12})_0,\ (s_{12})_0)$, and at least one of the pair of exponents $((r_{23})_0,\ (s_{23})_0)$, must equal one. Under these conditions, the argument is not essentially changed when all of the exponents are set equal to one, because replacement of the prime factors in any of the coefficients ρ_{12}, χ_{12}, ρ_{23} and χ_{23} by 1 only eliminates the presence of these prime factors from the remainder of the proof. The non-triviality of both fractions, therefore, can be included by setting $(r_{12})_0 = (r_{23})_0 = \{1\}$ and $(s_{12})_0 = (s_{23})_0 = \{1\}$. The expression (32) then would be $\frac{4k+1}{2}$ times the square of a rational number if

$$\begin{aligned} a_2 &= (4k+1)\rho_{12}\cdot\rho_{23}\cdot\frac{p^2}{2} \qquad b_2 = \chi_{12}\cdot\chi_{23}\cdot q^2 \\ &\qquad\qquad or \\ a_2 &= (4k+1)\chi_{12}\cdot\chi_{23}\cdot\frac{p^2}{2} \qquad b_2 = \rho_{12}\cdot\rho_{23}\cdot q^2 \end{aligned} \tag{33}$$

where $gcd(p,q) = 1$. If $a_2 = (4k+1)\rho_{12}\rho_{23}\frac{p^2}{2}$ and $b_2 = \chi_{12}\chi_{23}q^2$,

$$\begin{aligned} \frac{a_3}{b_3} &= \frac{a_2}{b_2}\frac{\rho_{23}}{\chi_{23}} = \frac{(4k+1)\rho_{12}\rho_{23}^2p^2}{2\chi_{12}\chi_{23}^2q^2} \\ \frac{a_3}{b_3}\frac{2\chi_{12}}{(4k+1)\rho_{12}} &= \frac{(\rho_{23}p)^2}{(\chi_{23}q)^2} \end{aligned} \tag{34}$$

* $\square$ denotes the square of a rational number.

Since $gcd(a_3, b_3) = 1$, the square-free factors can be separated in the fraction $\frac{a_3}{b_3} = \frac{\hat{a}_3}{\hat{b}_3} \cdot \frac{\hat{p}^2}{\hat{q}^2}$,

$$\frac{2\chi_{12}}{(4k+1)\rho_{12}} \frac{\hat{a}_3}{\hat{b}_3} = \frac{(\rho_{23} p\hat{q})^2}{(\chi_{23} q\hat{p})^2} \tag{35}$$

Since a_3 is even, and $\hat{a}_3$ is divisible by a single factor of 2, $\chi_{12} = \rho_{23}\frac{p}{2}\hat{q}$, and similarly, because $\hat{b}_3$ is odd, $\rho_{12} = \frac{1}{4k+1}\chi_{23} q\hat{p}$. Since $\frac{\rho_{12}\rho_{23}}{\chi_{12}\chi_{23}} = \frac{2}{4k+1}\frac{q\hat{p}}{p\hat{q}}$, rationality of $\left[\frac{2}{4k+1}\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right]^{\frac{1}{2}}$ also could be achieved by setting $a_2 = (4k+1)q\hat{p}\frac{p'^2}{2}$ and $b_2 = p\hat{q}q'^2$. Then

$$\begin{aligned} \frac{a_2}{b_2} \cdot \frac{\rho_{23}}{\chi_{23}} &= \frac{4k+1}{2} \frac{q\hat{p}p'^2}{p\hat{q}q'^2} \frac{\rho_{23}}{\chi_{23}} \\ &= \frac{\rho_{12}}{\chi_{12}} \cdot \frac{((4k+1)\rho_{23}\frac{p'}{2})^2}{(\chi_{23} q'^2)^2} \end{aligned} \tag{36}$$

Separating the square factors in $\frac{a_2}{b_2} = \frac{\hat{a}_2}{\hat{b}_2} \cdot \frac{\hat{p}'^2}{\hat{q}'^2}$, it follows that

$$\frac{\hat{a}_2}{\hat{b}_2} \frac{((4k+1)\frac{p}{2}\hat{q})}{(q\hat{p})} = \frac{((4k+1)\frac{p'}{2}\hat{q}')^2}{(q'\hat{p}')^2} \tag{37}$$

Either there is an overlap between the prime factors of $(4k+1)\frac{p}{2}$ and $\hat{q}$ or $\hat{a}_2 = (4k+1)\frac{p}{2}\hat{q} = (4k+1)\frac{p'}{2}\hat{q}'$, and similarly, there is either an overlap between the prime factors of q and $\hat{p}$ or $\hat{b}_2 = q\hat{p} = q'\hat{p}'$. Removing any overlap, then the remaining square factors can be separated in a_2 and b_2 obtaining the form $\frac{\hat{a}_2}{\hat{b}_2}$ for the square-free part of the ratio $\frac{a_2}{b_2}$. The equalities containing $\hat{a}_2$ and $\hat{b}_2$ imply that $\hat{p} > \hat{p}' \geq p' > p$ and $\hat{q} > \hat{q}' \geq q' > q$. By interchanging the roles of a_2, b_2 and a_3, b_3 in the above argument, the inequalities $p > \hat{p}$ and $q > \hat{q}$ can be derived, implying a contradiction. Thus, when $\ell = 3$, it should not be possible to find coefficients $\{a_i\}$ and $\{b_i\}$ satisfying equation (25) such that $\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}$ is $\frac{4k+1}{2}$ times the square of a rational number. The validity of this result is confirmed by the non-existence of odd perfect numbers with four different prime factors.

A variation of the standard induction argument can be used to show that there cannot be different odd perfect numbers with prime decompositions $(4k+1)^{4m+1}\prod_{i=1}^{\ell-1} q_i^{2\alpha_i}$ and $(4k'+1)^{4m'+1}\prod_{i=1}^{\ell} q_i'^{2\alpha_i'}$.

When ℓ is odd,

$$\frac{q_1^{2\alpha_1+1}-1}{q_1-1} \cdots \frac{q_{\ell-1}^{m_{\ell-1}}-1}{q_{\ell-1}-1} = \frac{b_1 \ldots b_{\ell-1}}{a_1 \ldots a_{\ell-1}} \left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\ell-1} = \frac{b_1 \ldots b_{\ell-1}}{a_1 \ldots a_{\ell-1}} \cdot \Box \tag{38}$$

rationality of square root of the product of repunits with $\ell-1$ prime bases $\{q_i, i = 1, ..., \ell-1\}$ would require

$$\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\rho_\ell \frac{b_\ell}{a_\ell} \cdot \Box \tag{39}$$

and

$$\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\rho_\ell \cdot \Box \tag{40}$$

Since the values $q_\ell = 3$ and $\alpha_\ell = 2$ can be excluded from the product of repunits, ρ_ℓ is odd and does not equal 1, so that $\frac{b_1...b_\ell}{a_1...a_\ell} \neq 2(4k+1)\Box$. The square root of the product of repunits with ℓ prime bases $\{q_i, i = 1, ..., \ell\}$ is therefore not rational.

When ℓ is even,

$$\frac{q_1^{2\alpha_1+1}-1}{q_1-1} ... \frac{q_{\ell-1}^{2\alpha_{\ell-1}+1}-1}{q_{\ell-1}-1} = \frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}} \left(\frac{(4k+1)^{4m+2}-1}{4k}\right) \cdot \Box \tag{41}$$

so that rationality of the square root expression with $\ell-1$ primes $\{q_i,\ i = 1, ..., \ell-1\}$ requires

$$\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\rho_\ell \cdot \left(\frac{(4k+1)^{4m+2}-1}{4k}\right) \cdot \Box \tag{42}$$

Again, since $\rho_\ell \neq 1$, equation (42) implies that $\frac{b_1...b_\ell}{a_1...a_\ell}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right) \neq 2(4k+1)\Box$ or equivalently that the square root expression with ℓ primes $\{q_i,\ i = 1, ..., \ell\}$ is not rational.

The proof can be continued for $\ell > 3$ by assuming that there do not exist any odd primes $q_1, ..., q_{\ell-1}$ and $4k+1$ such that $\sqrt{2(4k+1)}\left[\frac{q_1^{2\alpha_1+1}-1}{q_1-1}...\frac{q_\ell^{2\alpha_\ell+1}-1}{q_{\ell-1}-1}\right]^{\frac{1}{2}}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\frac{1}{2}}$ is rational and proving that the same property is valid when ℓ odd primes $q_1, ..., q_\ell$ arise in the prime decomposition of the integer N.

If ℓ is odd, $\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\frac{(\ell+1)}{2}}$ is integer, and non-existence of odd perfect numbers of the form $(4k+1)^{4m+1}q_1^{2\alpha_1}...q_{\ell-1}^{2\alpha_{\ell-1}}$ is equivalent to the condition $\frac{b_1..b_\ell}{a_1...a_\ell} \neq 2(4k+1)\Box$. Since

$$2(4k+1)\frac{q_1^{m_1}-1}{q_1-1}...\frac{q_{l-1}^{m_{\ell-1}-1}-1}{q_{\ell-1}-1}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right) = 2(4k+1)\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}} \cdot \left(\frac{(4k+1)^{4m+2}-1}{4k}\right) \cdot \Box \tag{43}$$

Since the irrationality of the square root expression is assumed to hold generally for $\ell-1$ odd primes $\{q_i\}$ and any value of $4k+1$, the effect of the inclusion of another prime q_ℓ

can be deduced. Thus, given an arbitrary set of ℓ odd primes, q_1, ..., q_ℓ and some prime of the form $4k+1$, irrationality of the square root of expression (43) implies that

$$\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}} \neq 2(4k+1) \left(\frac{(4k+1)^{4m+2}-1}{4k}\right) \cdot \Box \tag{44}$$

However, by equation (25), $\left(\frac{(4k+1)^{4m+2}-1}{4k}\right) = \frac{a_\ell}{b_\ell}\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$, and if $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \equiv \rho_\ell \chi_\ell^2$, separating the square-free factors from the factors with even exponents. it follows that

$$\begin{aligned} \frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}} &\neq 2(4k+1)\ \rho_\ell \frac{a_\ell}{b_\ell} \cdot \Box \\ \frac{b_1...b_\ell}{a_1...a_\ell} &\neq 2(4k+1)\ \rho_\ell \cdot \Box \end{aligned} \tag{45}$$

The form of the relation (45) is valid for arbitrary values of $\frac{b_\ell}{a_\ell}$, but the choice of ρ_ℓ is specific to the repunit $\frac{{q_\ell}^{2\alpha_\ell+1}-1}{q_\ell-1}$. Since $\frac{{q_\ell}^{2\alpha_\ell+1}-1}{q_\ell-1}$ is the square of an integer only when $q_\ell = 3,\ \alpha_\ell = 2$, it is preferable to represent the rationality condition for $\ell-1$ and ℓ primes $\{q_i\}$ as

$$\begin{aligned} \frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}} &= 2(4k+1)\ \omega_{\ell-1}\rho_\ell \frac{a_\ell}{b_\ell} \cdot \Box \\ \frac{b_1...b_\ell}{a_1...a_\ell} &= 2(4k+1)\ \omega_\ell \cdot \Box \end{aligned} \tag{46}$$

when ℓ is odd. Irrationality of the square root expression for $\ell-1$ primes $\{q_i,\ i=1,...\ell-1\}$, which requires that $\omega_{\ell-1} \neq 1$ is a square-free integer, implies irrationality for ℓ primes $\{q_i, i=1,...,\ell\}$ if $\omega_{\ell-1}\rho_\ell = \omega_\ell \neq 1$ is square-free.

When ℓ is even, odd perfect numbers of the form $(4k+1)^{4m+1}q_1^{2\alpha_1}...q_{\ell-1}^{2\alpha_{\ell-1}}$ do not exist if $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}} \neq 2(4k+1)\cdot\Box$. Then $\frac{b_1...b_\ell}{a_1...a_\ell}\cdot\left(\frac{(4k+1)^{4m+2}-1}{4k}\right) \neq 2(4k+1)\rho_\ell\cdot\Box$. Irrationality of the square root expression with $\ell-1$ primes $\{q_i, i=1...,\ell-1\}$ also can be represented as as $\frac{b_1...b_{\ell-1}}{a_1...a_{\ell-1}} = 2(4k+1)\omega_{\ell-1}\cdot\Box$ where $\omega_{\ell-1}\neq 1$ is a square-free number. Consequently, $\frac{b_1...b_\ell}{a_1...a_\ell} = 2(4k+1)\omega_{\ell-1}\frac{b_\ell}{a_\ell}\cdot\Box$. Since irrationality of the square root expression with ℓ primes $\{q_i, i=1,...\ell\}$ would equivalent to

$$\begin{aligned} \frac{b_1...b_\ell}{a_1...a_\ell}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right) &= 2(4k+1)\ \omega_\ell\cdot\Box \\ \frac{b_1...b_\ell}{a_1...a_\ell} &= 2(4k+1)\ \omega_\ell\rho_\ell\frac{a_\ell}{b_\ell}\cdot\Box \end{aligned} \tag{47}$$

this again can be achieved if $\omega_{\ell-1}\rho_\ell = \omega_\ell\cdot\Box$.

For any prime divisor p

$$\begin{aligned}
v_p(\omega_{\ell-1}) \;&=\; \sum_{i=1}^{\ell-1} \left[v_p\left(\frac{q_i^{e_i}-1}{q_i-1}\right) \delta\left(\frac{n_i}{e_i} - \left[\frac{n_i}{e_i}\right]\right) \;+\; v_p(n_i) \right] \\
&\qquad\qquad +\; v_p\left(\frac{(4k+1)^{4m+2}-1}{4k}\right) \qquad\qquad (mod\ 2) \\
v_p(\omega_\ell) \;&=\; \sum_{i=1}^{\ell} \left[v_p\left(\frac{q_i^{e_i}-1}{q_i-1}\right) \delta\left(\frac{n_i}{e_i} - \left[\frac{n_i}{e_i}\right]\right) \;+\; v_p(n_i) \right] \\
&\qquad\qquad +\; v_p\left(\frac{(4k+1)^{4m+2}-1}{4k}\right) \qquad\qquad (mod\ 2)
\end{aligned} \tag{48}$$

where $e_i = ord_p(q_i)$. It follows that

$$v_p(\omega_\ell) \;=\; v_p(\omega_{\ell-1}) \;+\; v_p\left(\frac{q_\ell^{e_\ell}-1}{q_\ell-1}\right) \;+\; v_p(n_\ell) \tag{49}$$

Suppose that p is one of the extra prime divisors so that $v_p(\omega_{\ell-1}) = 1$. If $e_\ell \nmid n_\ell$ or $p \nmid n_\ell$, then $p \not\Big| \frac{{q_\ell}^{n_\ell}-1}{q_\ell-1}$ and $v_p(\omega_\ell) = 1$.

If $p^h \Big\| \frac{q_\ell^{e_\ell}-1}{q_\ell-1}$, and p is a primitive prime divisor of this repunit, then $v_p(n_\ell) = 0$ and $v_p(\omega_\ell) = 1+h\ (mod\ 2)$. Since $v_p(\omega_\ell) = 0\ (mod\ 2)$ if $h = 1$, it would be the next category of prime divisors, with the property $v_p\left(\frac{q_\ell^{e_\ell}-1}{q_\ell-1}\right) = 2$ or equivalently $Q_{q_\ell} \equiv 0\ (mod\ p)$, which contributes non-trivially to a square-free coefficient ω_ℓ.

Since it has been assumed that the square root expression with $\ell-1$ primes $\{q_i, i = 1, ..., \ell-1\}$ is irrational, there is either an unmatched primitive divisor or an imprimitive divisor in the product $\prod_{i=1}^{\ell-1} \frac{q_i^{2\alpha_i+1}-1}{q_i-1} \cdot \frac{(4k+1)^{4m+2}-1}{4k}$. Suppose that the extra prime divisor $\hat{p}_j$ is a factor of the repunit $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. By equation (27),

$$\frac{q_j^{2\alpha_j+1}-1}{q_j-1} \;=\; \rho_j\ \chi_j^2 \;=\; \frac{b_{j\ell}}{a_{j\ell}} \frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1} \;=\; \frac{b_{j\ell}}{a_{j\ell}}\ \rho_\ell\ \chi_\ell^2 \tag{50}$$

so that $\rho_j\rho_\ell = \frac{b_{j\ell}}{a_{j\ell}} \cdot \square$.

To proceed further, it is first useful to choose the exponent $2\alpha_\ell+1$ to be equal to $2\alpha_j+1$. If $p|(q_j-1)$, $p^{\hat{h}_j}|(2\alpha_j+1)$, $p|(q_\ell-1)$, $p^{\hat{h}_\ell}|(2\alpha_\ell+1)$, then $p^{\hat{h}_j}\Big|\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $p^{\hat{h}_\ell}\Big|\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$

When $\alpha_j = \alpha_\ell$, $p^{h_j} = p^{\hat{h}_j} = p^{\hat{h}_\ell} = p^{h_\ell}$, where h_j and h_ℓ denote the exponents of p exactly dividing the repunits with bases q_j and q_ℓ respectively, so that this prime divisor will be absorbed into the square factors.

If $p|(q_j - 1)$ and $p \nmid (q_\ell - 1)$, then $h_j = \hat{h}_j$ and $h_\ell = \hat{h}_\ell + v_p\left(\frac{q_\ell^{e_\ell}-1}{q_\ell - 1}\right)$. Since $\hat{h}_j = \hat{h}_\ell$ when $\alpha_j = \alpha_\ell$, $h_\ell = h_j + v_p\left(\frac{q_\ell^{e_\ell}-1}{q_\ell-1}\right)$. Matching of the prime factors in the two repunits would require $v_p\left(\frac{q_\ell^{e_\ell}-1}{q_\ell-1}\right) = 0\ (mod\ 2)$. Because $p|(q_\ell^{e_\ell}-1)$, the minimum value of this exponent is 2, implying that $Q_{q_\ell} \equiv 0\ (mod\ p)$. Conversely, if $Q_{q_\ell} \not\equiv 0\ (mod\ p)$ or $Q_{q_\ell} \equiv 0\ (mod\ p^{h'_\ell - 1})$, $Q_{q_\ell} \not\equiv 0\ (mod\ p^{h'_\ell})$, where h'_ℓ is odd, the prime divisor p in the product of the two repunits cannot be entirely absorbed into the square factors. Similar conclusions hold when $p \nmid (q_j - 1)$ and $p|(q_\ell - 1)$.

Let p be an imprimitive prime divisor such that $p \nmid (q_j - 1)$ and $p \nmid (q_\ell - 1)$, then $v_p\left(\frac{q_j^{n_j}-1}{q_j-1}\right) = v_p(q_j^{n_j} - 1)$ and $v_p\left(\frac{q_\ell^{n_\ell}-1}{q_\ell-1}\right) = v_p(q_\ell^{n_\ell} - 1)$. If $p^h|n_\ell$, and $n_j = n_\ell$, then $h_j = h_\ell = h$, again implying that the prime divisor can be absorbed into the square factors.

The arithmetic primitive factors of $\frac{q_j^{n_j}-1}{q_j-1}$ and $\frac{q_\ell^{n_\ell}-1}{q_\ell-1}$, $\frac{\Phi_{n_j}(q_j)}{p_j}$ and $\frac{\Phi_{n_\ell}(q_\ell)}{p_\ell}$ respectively, are different when $n_j = n_\ell$, except possibly for solutions generated by the prime equation $\frac{q_\ell^n - 1}{q_j^n - 1} = p$ required when either $p_j = gcd(n_j, \Phi_{n_j}(q_j))$ or $p_\ell = gcd(n_\ell, \Phi_{n_\ell}(q_\ell))$ equals 1. The algebraic primitive factors $\Phi_{n_j}(q_j)$ and $\Phi_{n_\ell}(q_\ell)$ necessarily will be different if $n_j = n_\ell$. Consider a prime divisor p' of the arithmetic primitive factors which is raised to a different power in $\frac{\Phi_{n_j}(q_j)}{p_j}$ and $\frac{\Phi_{n_\ell}(q_\ell)}{p_\ell}$. If this prime is the only factor with this property, then $\frac{q_\ell^{n_\ell}-1}{q_j^{n_j}-1} = \frac{q_\ell^{n_\ell}-1}{q_j^{n_\ell}-1} = (p')^{h_\ell - h_j}$, and the non-existence of solutions to this equation for $h_\ell - h_j \geq 2$ has been shown in §4.

The error in the approximation is given by $\frac{q_\ell^{n_\ell}}{q_j^{n_\ell}}\left[1 - \frac{1}{q_\ell^{n_\ell}} + \frac{1}{q_j^{n_j}} + \mathcal{O}\left(\frac{1}{q_j^{n_\ell} q_\ell^{n_\ell}}\right)\right]$, and since $\left|\frac{1}{q_j^{n_\ell}} - \frac{1}{q_\ell^{n_\ell}}\right| < min\left(\frac{1}{q_j^{n_\ell}}, \frac{1}{q_\ell^{n_\ell}}\right)$, the error is less than $\frac{q_\ell^{n_\ell}}{q_j^{n_\ell}(q_j^{n_\ell}-1)} \simeq \frac{q_\ell^{n_\ell}}{q_j^{2n_\ell}}$. Given a rational number $\frac{a}{b}$, the inequality $\left|\frac{a}{b} - \frac{z_2}{z_1}\right| < \frac{1}{z_1^2}$ has a finite number of solutions satisfying $z_1 < b$, $gcd(z_1, z_2) = 1$ [39]. In particular, solutions to

$$\left|\frac{q_\ell{}^{n_\ell}}{q_j^{n_\ell}} - \frac{z_2}{z_1}\right| = \left|\frac{q_\ell^{n_\ell}}{q_j^{n_\ell}} - \frac{y_2^2}{y_1^2}\right| < \frac{1}{y_1^4} \tag{51}$$

will be constrained by the inequality $y_1 < q_j^{\frac{n_\ell}{2}}$. The condition $\left|\frac{q_\ell^{n_\ell}}{q_j^{n_\ell}} - \frac{y_2^2}{y_1^2}\right| < \frac{q_\ell^{n_\ell}}{q_j^{2n_\ell}}$ satisfied when $\frac{q_j^{\frac{n_\ell}{2}}}{q_\ell^{\frac{n_\ell}{4}}} < y_1 < q_j^{\frac{n_\ell}{2}}$.

Since it has been established that square classes of the repunits $\frac{q^n-1}{q-1}$ consist of only one element [40], it follows that $(q_\ell^{n_{\ell_1}} - 1)(q_\ell^{n_{\ell_2}} - 1) = (\kappa')^2(q_\ell - 1)^2(y_1')^2(y_2')^2$ and there is only one representative from each sequence $\{q_j^{n_j} - 1,\ n_j \in \mathbb{Z}\}$, $\{q_\ell^{n_\ell} - 1,\ n_\ell \in \mathbb{Z}\}$ which has a specified square-free factor κ. Thus, $\frac{q_\ell^{n_\ell}-1}{q_j^{n_\ell}-1} \neq \frac{y_2^2}{y_1^2}$ unless $n_{q_j}(\kappa)$ coincides with $n_{q_\ell}(\kappa)$. If $q_\ell^{n_\ell} - 1 = \kappa(y_2')^2$ and $q_j^{n_\ell} - 1 = \kappa(y_1')^2$, and $\frac{y_2^2}{y_1^2}$ is the irreducible form of $\frac{(y_2')^2}{(y_1')^2}$, it follows that $y_1 < \frac{q_j^{\frac{n_\ell}{2}}}{\sqrt{\kappa\hat{\kappa}^2}}$, where $\hat{\kappa} = gcd(y_1', y_2')$. Both inequalities for y_1 cannot be satisfied if $q_\ell^{\frac{n_\ell}{4}} < \sqrt{\kappa\hat{\kappa}^2}$ or equivalently $q_\ell^{\frac{n_\ell}{2}} < gcd(q_j^{n_\ell} - 1, q_\ell^{n_\ell} - 1)$. When the pair of primes (q_j, q_ℓ) satisfies the last inequality, the prime divisors in ρ_j and ρ_ℓ do not match and the product of the repunits $\frac{q_j^{n_j}-1}{q_j-1}$ and $\frac{q_\ell^{n_\ell}-1}{q_\ell-1}$, with $n_j \neq n_\ell$, is not a perfect square.

The number of solutions to the inequality $|ax^n - by^n| \leq h$ when $x \geq \left(\frac{2h}{a^{1-\rho}\alpha}\right)^{\frac{1}{\frac{n}{2}-1}}$ with $\alpha = \left(\frac{b}{a}\right)^{\frac{1}{n}}$ does not exceed $6 + \frac{1}{ln\frac{n}{2}}\left[29 + ln\rho^{-1} + ln\left(1 + \frac{ln\ 2h}{ln\ a}\right)\right]$ [41]. Setting $\frac{q_\ell^n-1}{q_j^n-1} \simeq \frac{y_2^2}{y_1^2}$, it follows that $y_1^2(q_\ell^n - 1) \simeq y_2^2(q_j^n - 1)$ leading to consideration of the inequality $|y_2^2 q_j^n - y_1^2 q_\ell^n| \leq |y_2^2 - y_1^2|$. The constraint placed on q_j is

$$q_j \geq \left(\frac{2|y_2^2 - y_1^2|}{y_2^{2(1-\rho)}\left(\frac{y_2^2}{y_1^2}\right)^{\frac{1}{n}}}\right)^{\frac{1}{\frac{n}{2}-1}} \tag{52}$$

Since $\frac{q_j-1}{q_\ell-1} \geq \left(\frac{y_1^2}{y_2^2}\right)^{\frac{1}{n}} \geq \frac{q_j}{q_\ell}$, it is sufficient for q_j to satisfy the stronger constraint

$$q_j \geq \left(2\frac{q_\ell - 1}{q_j - 1}y_2^{2\rho}\right)^{\frac{1}{\frac{n}{2}-1}} \tag{53}$$

which is equivalent to an upper bound for y_2 of

$$y_2^2 \geq q_j^{\rho^{-1}\left(\frac{n}{2}-1\right)} \cdot \left(\frac{1}{2}\frac{q_j - 1}{q_\ell - 1}\right)^{\rho^{-1}} \tag{54}$$

This condition defines an allowable range of values for y_2 when $\rho \leq \frac{1}{2}$. The number of solutions to the inequality is not greater than

$$\begin{aligned} 6 \ + \ \frac{1}{ln\ (\frac{n}{2})}\left(29 \ + \ ln\ \rho^{-1} \ + \ ln\left(1 \ + \ \frac{ln\ (2|y_2^2 - y_1^2|)}{ln\ y_2^2}\right)\right) \\ \leq \ 6 \ + \ \frac{29 \ + \ ln(2 + ln2) \ + \ ln\ \rho^{-1}}{ln\ \frac{n}{2}} \end{aligned} \tag{55}$$

Any adjustment in n_ℓ will introduce additional prime divisors. Either they shall be new prime factors of the exponent or primitive divisors [42]-[45]. If n_ℓ is multiplied by a prime factor $\hat{p}^{r_\ell}$, where $\hat{p}|\rho_\ell$, then the product $\rho_\ell \hat{p}^{r_\ell}$ will contain the power $\hat{p}_\ell^{1+r_\ell}$. While the prime power can be absorbed into the product of square factors only when r_ℓ is odd, the repunit $\frac{q_\ell^{n_\ell \hat{p}^{r_\ell}}-1}{q_\ell-1}$ has extra primitive divisors, giving rise to a non-trivial ω_ℓ, implying irrationality of the square root expression with ℓ primes $\{q_i,\ i=1,...\ell\}$. Moreover, $gcd(\Phi_{\hat{p}^i}(q), \Phi_{\hat{p}^j}(q)) = 1$ when $i \neq j$ and $p \nmid (q-1)$, multiplication of the index by $\hat{p}^{r_\ell}$ will introduce new prime divisors through the decomposition of the repunit $\frac{q_\ell^{n_\ell \hat{p}^{r_\ell}}-1}{q_\ell-1} = \prod_{\substack{d|n_\ell \hat{p}^{r_\ell} \\ d>1}} \Phi_d(q_\ell)$.

The abstract argument given for $\ell = 3$ could also be extended to higher values of ℓ. This approach would consist of the demonstration of the property $\frac{b_1...b_\ell}{a_1...a_\ell} \cdot \neq 2(4k+1) \cdot \square$ if ℓ is odd, and $\frac{b_1...b_\ell}{a_1...a_\ell} \cdot \neq 2(4k+1)\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)$ if ℓ is even, given that there are no sets of primes $\{q_i\}$ with less than ℓ elements satisfying the rationality condition. It may be noted that since

$$\begin{aligned}
\frac{b_1...b_\ell}{a_1...a_\ell} &= \left(\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right)\left(\frac{b_{46}}{a_{46}}\frac{a_5}{b_5}\right)...\left(\frac{b_{\ell-2,\ell}}{a_{\ell-2,\ell}}\frac{a_{\ell-1}}{b_{\ell-1}}\right)\cdot\left(\frac{b_2b_3b_5b_6...b_{\ell-1}b_\ell}{a_2a_3a_5a_6...a_{\ell-1}a_\ell}\right)^2 \\
&\qquad\qquad when\ \ell \equiv\ 0\ (mod\ 3) \\
\frac{b_1...b_\ell}{a_1...a_\ell} &= \left(\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right)\left(\frac{b_{46}}{a_{46}}\frac{a_5}{b_5}\right)...\left(\frac{b_{\ell-3,\ell-1}}{a_{\ell-3,\ell-1}}\frac{a_{\ell-2}}{b_{\ell-2}}\right)\frac{b_\ell}{a_\ell}\cdot\left(\frac{b_2b_3b_5b_6...b_{\ell-2}b_{\ell-1}}{a_2a_3a_5a_6...a_{\ell-2}a_{\ell-1}}\right)^2 \\
&\qquad\qquad when\ \ell \equiv\ 1\ (mod\ 3) \\
\frac{b_1...b_\ell}{a_1...a_\ell} &= \left(\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right)\left(\frac{b_{46}}{a_{46}}\frac{a_5}{b_5}\right)...\left(\frac{b_{\ell-4,\ell-2}}{a_{\ell-4,\ell-2}}\frac{a_{\ell-3}}{b_{\ell-3}}\right)\frac{b_{\ell-1}b_\ell}{a_{\ell-1}a_\ell}\cdot\left(\frac{b_2b_3b_5b_6...b_{\ell-3}b_{\ell-2}}{a_2a_3a_5a_6...a_{\ell-3}a_{\ell-2}}\right)^2 \\
&\qquad\qquad when\ \ell \equiv\ 2\ (mod\ 3)
\end{aligned} \tag{56}$$

and $\frac{b_{13}}{a_{13}}\frac{a_2}{b_2} = 2(4k+1)\frac{\bar{\rho}_1}{\bar{\chi}_1}\cdot\square$, ..., $\frac{b_{\ell-k'-2,\ell-k'}}{a_{\ell-k'-2,\ell-k'}} = 2(4k+1)\frac{\bar{\rho}_{\left[\frac{\ell}{3}\right]}}{\bar{\chi}_{\left[\frac{\ell}{3}\right]}}\cdot\square$, where $\ell \equiv k'\ (mod\ 3)$, $k' = 0, 1, 2$, $\bar{\rho}_1$,..., $\bar{\rho}_{\left[\frac{\ell}{3}\right]}$, $\bar{\chi}_1$,..., $\bar{\chi}_{\left[\frac{\ell}{3}\right]}$ are square-free factors, the quotient will equal $(2(4k+1))^{\left[\frac{\ell}{3}\right]} f_{k'}\frac{\bar{\rho}_1}{\bar{\chi}_1}...\frac{\bar{\rho}_{\left[\frac{\ell}{3}\right]}}{\bar{\rho}_{\left[\frac{\ell}{3}\right]}}\cdot\square$ with $f_0 = 1$, $f_1 = \frac{b_\ell}{a_\ell}$ and $f_2 = \frac{b_{\ell-1}b_\ell}{a_{\ell-1}a_\ell}$. It has been established that $\frac{b_\ell}{a_\ell} \neq 2(4k+1)\cdot\square$ because there is no odd integer of the form $(4k+1)^{4m+1}q_\ell^{2\alpha_\ell}$ which satisfies the relation $\frac{\sigma(N)}{N} = 2$. $\frac{b_{\ell-1}b_\ell}{a_{\ell-1}a_\ell} \neq 2(4k+1)\left(\frac{4k+1)^{4m+2}-1}{4k}\right)\cdot\square$ because of the non-existence of odd perfect numbers of the type $(4k+1)^{4m+1}q_{\ell-1}^{2\alpha_{\ell-1}}q_\ell^{2\alpha_\ell}$. Setting

$\frac{b_\ell}{a_\ell} = 2(4k+1)\frac{\hat{\rho}_{\ell 1}}{\hat{\chi}_1}\cdot\Box$ and $\frac{b_{\ell-1}b_\ell}{a_{\ell-1}a_\ell} = 2(4k+1)\frac{\hat{\rho}_{\ell 2}}{\hat{\chi}_{\ell 2}}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\cdot\Box$, it follows that

$$\begin{aligned}
\frac{b_1...b_\ell}{a_1...a_\ell} &= (2(4k+1))^{\frac{\ell}{3}}\frac{\bar{\rho}_1}{\bar{\chi}_1}...\frac{\bar{\rho}_{\frac{\ell}{3}}}{\bar{\chi}_{\frac{\ell}{3}}}\cdot\Box && \ell\equiv\ 0\ (mod\ 3)\\
\frac{b_1...b_\ell}{a_1...a_\ell} &= (2(4k+1))^{[\frac{\ell}{3}]+1}\frac{\bar{\rho}_1}{\bar{\chi}_1}...\frac{\bar{\rho}_{[\frac{\ell}{3}]}}{\bar{\chi}_{[\frac{\ell}{3}]}}\cdot\frac{\hat{\rho}_{\ell 1}}{\hat{\chi}_{\ell 1}}\cdot\Box && \ell\equiv\ 1\ (mod\ 3)\\
\frac{b_1...b_\ell}{a_1...a_\ell} &= (2(4k+1))^{[\frac{\ell}{3}]+1}\frac{\bar{\rho}_1}{\bar{\chi}_1}...\frac{\bar{\rho}_{[\frac{\ell}{3}]}}{\bar{\chi}_{[\frac{\ell}{3}]}}\cdot\frac{\hat{\rho}_{\ell 2}}{\hat{\chi}_{\ell 2}}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\cdot\Box && \ell\equiv\ 2\ (mod\ 3)
\end{aligned}\tag{57}$$

and the coefficients $\{a_i, b_i\}$ will not satisfy the rationality condition when the square-free factors $\bar{\rho}_1, ..., \bar{\rho}_{[\frac{\ell}{3}]}, \hat{\rho}_{\ell 1}, \hat{\rho}_{\ell 2}, \bar{\chi}_1, ..., \bar{\chi}_{[\frac{\ell}{3}]}, \hat{\chi}_{\ell 1}, \hat{\chi}_{\ell 1}$ have prime divisors other than 2 and $4k+1$ which do not match to produce the square of a rational number.

When ℓ is odd and greater than 5, there always exists an odd integer ℓ_o and an even integer ℓ_e such that $\ell = 3\ell_o + 2\ell_e$, implying the following identity

$$\frac{b_1...b_\ell}{a_1...a_\ell} = \left(\frac{b_{13}}{a_{13}}\frac{a_2}{b_2}\right)\left(\frac{b_{46}}{a_{46}}\frac{a_5}{b_5}\right)...\left(\frac{b_{3\ell_o-2,3\ell_o}}{a_{3\ell_0-2,3\ell_o}}\frac{a_{3\ell_o-1}}{b_{3\ell_o-1}}\right)\left(\frac{b_{3\ell_0+1}b_{3\ell_0+2}}{a_{3\ell_0+1}a_{3\ell_0+2}}\right)...\left(\frac{b_{\ell-1}b_\ell}{a_{\ell-1}a_\ell}\right)\cdot\Box \tag{58}$$

Consequently,

$$\begin{aligned}
\frac{b_1...b_\ell}{a_1...a_\ell} &= (2(4k+1))^{\ell_0+\ell_e}\ \cdot\ \frac{\bar{\rho}_1}{\bar{\chi}_1}...\frac{\bar{\rho}_{\ell_o}}{\bar{\chi}_{\ell_o}}...\frac{\hat{\rho}_{\ell-2\ell_e+2,2}}{\hat{\chi}_{\ell-2\ell_e+1,2}}...\frac{\hat{\rho}_{\ell 2}}{\hat{\chi}_{\ell 2}}\ \cdot\ \left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\ell_e}\ \cdot\ \Box\\
&= 2(4k+1)\ \cdot\ \frac{\bar{\rho}_1}{\bar{\chi}_1}...\frac{\bar{\rho}_{\ell_o}}{\bar{\chi}_{\ell_o}}...\frac{\hat{\rho}_{\ell-2\ell_e+2,2}}{\hat{\chi}_{\ell-2\ell_e+2,2}}...\frac{\hat{\rho}_{\ell 2}}{\hat{\chi}_{\ell 2}}\ \cdot\ \Box
\end{aligned}\tag{59}$$

Regardless of the factors of 2 and $4k+1$, the coefficients $\{a_i, b_i\}$ will produce an irrational square root expression (26) for odd ℓ if the product of fractions $\prod_{i=1}^{\ell_o}\frac{\bar{\rho}_i}{\bar{\chi}_i}\prod_{j=1}^{\ell_e}\frac{\hat{\rho}_{\ell-2j+2,2}}{\hat{\chi}_{\ell-2j+2,2}}$ is not the square of a rational number.

If ℓ is even and greater than 4, there always exists an odd integer ℓ_o and an even integer ℓ_e such that $\ell = 2\ell_o + 3\ell_e$. From the identity

$$\frac{b_1...b_\ell}{a_1...a_\ell} = \left(\frac{b_1b_2}{a_1a_2}\right)...\left(\frac{b_{2\ell_o-1}b_{2\ell_o}}{a_{2\ell_o-1}a_{2\ell_o}}\right)...\left(\frac{b_{2\ell_o+1,2\ell_0+3}}{a_{2\ell_o+1}a_{2\ell_o+3}}\frac{a_{2\ell_o+2}}{b_{2\ell_o+2}}\right)...\left(\frac{b_{\ell-2,\ell}}{a_{\ell-2,\ell}}\frac{a_{\ell-1}}{b_{\ell-1}}\right)\cdot\Box \tag{60}$$

it follows that

$$\begin{aligned}
\frac{b_1...b_\ell}{a_1...a_\ell} &= (2(4k+1))^{\ell_o+\ell_e}\left(\frac{(4k+1)^{4m+2}-1}{4k}\right)^{\ell_o}\ \cdot\ \frac{\hat{\rho}_{22}}{\hat{\chi}_{22}}...\frac{\hat{\rho}_{2\ell_o,2}}{\hat{\chi}_{2\ell_o,2}}\frac{\bar{\rho}_{\ell-3\ell_e+1}}{\bar{\chi}_{\ell-3\ell_e+1}}...\frac{\bar{\rho}_{\ell-2}}{\bar{\chi}_{\ell-2}}\ \cdot\ \Box\\
&= 2(4k+1)\ \cdot\ \frac{\hat{\rho}_{22}}{\hat{\chi}_{22}}...\frac{\hat{\rho}_{2\ell_o,2}}{\hat{\chi}_{2\ell_o,2}}\frac{\bar{\rho}_{\ell-3\ell_e+1}}{\bar{\chi}_{\ell-3\ell_e+1}}...\frac{\bar{\rho}_{\ell-2}}{\bar{\chi}_{\ell-2}}\ \cdot\ \left(\frac{(4k+1)^{4m+2}-1}{4k}\right)\ \cdot\ \Box
\end{aligned}\tag{61}$$

Again, the factors of 2 and $4k+1$ are not relevant, and the coefficients $\{a_i, b_i\}$ give rise to an irrational square root expression (26) for even ℓ if $\prod_{i=1}^{\ell_o} \frac{\hat{\rho}_{2i,2}}{\hat{\chi}_{2i,2}} \prod_{j=1}^{\ell_e} \frac{\bar{\rho}_{\ell-3j+1}}{\bar{\chi}_{\ell-3j+1}}$ is not the square of a rational number. $\blacksquare$

7. Conclusion

The rationality condition provides an analytic method for investigating the existence of odd perfect numbers. The aim of this approach then becomes the proof of the existence of an unmatched prime divisor in the product of the repunits, since the square root of any such divisor would be irrational, contrary to the condition for the existence of an odd perfect number. An upper bound for the density of odd integers greater than 10^{300}, in an interval of fixed length, which could satisfy $\frac{\sigma(N)}{N} = 2$, may be found by considering the square root expression containing the product of repunits, combining the estimate of the density of square-full numbers in this range with the probability of an integer being expressible as the product of repunits with prime bases multiplied by $2(4k+1)$. Repunits form a special class of Lucas sequences, and the properties of primitive and imprimitive prime divisors of these sequences can be used to determine the powers of primes dividing the product of repunits. A comparison of the divisibility properties of Lucas sequences $U_n(q+1, q)$ with different values of q has been undertaken in §4. Specifically, the arithmetic primitive factors of these repunits, products of the primitive prime power divisors, can be compared for different values of the prime basis, and it has been shown that they could only be equal if the indices of Lucas sequences differ, except possibly for pairs of divisors $\left(\Phi_n(q_i), \frac{\Phi_n(q_j)}{p_j}\right)$ generated by the prime equation $\frac{q_j^n - 1}{q_i^n - 1} = p$. In the second theorem, non-existence of the odd perfect numbers for a large set of primes $\{q_i, i = 1, ..., \ell;\ 4k+1\}$, exponents $\{2\alpha_i + 1, i = 1, ..., \ell;\ 4m+1\}$ and values of ℓ using the method of induction adapted to the coefficients $\{a_i, b_i\}$ in the product of repunits. An abstract argument is given for the non-existence of coefficients satisfying the rationality condition when $\ell = 3$ and then various results are proven for $\ell > 3$ by using the properties of prime divisors of product of two repunits, $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$ and $\frac{q_\ell^{2\alpha_\ell+1}-1}{q_\ell-1}$, belonging to each of the four categories: (i) $p|(q_j-1)$, $p|(q_\ell-1)$ (ii) $p|(q_j-1)$, $p \nmid (q_\ell-1)$ (iii) $p \nmid (q_j-1)$, $p|(q_\ell-1)$ (iv) $p \nmid (q_j-1)$, $p \nmid (q_\ell-1)$. Irrationality of the square root expression for any set of $\ell-1$ primes $\{q_i,\ i = 1, ..., \ell-1\}$ implies that each unmatched prime divisor in the product of repunits with bases $\{q_i,\ i = 1, ..., \ell-1, 4k+1\}$ can be associated with a single repunit, because factors of other repunits divisible by this prime contain powers of the prime with the exponent summing up to an even integer. Supposing, for example, that the repunit containing this extra prime divisor is $\frac{q_j^{2\alpha_j+1}-1}{q_j-1}$. The problem of determining whether this

prime divisor remains unmatched, when an additional prime q_ℓ in the decomposition of the odd integer N is included, depends on the feasibility of matching the prime divisors of each pair of repunits $(U_{n_j}(q_j+1, q_j), U_{n_\ell}(q_\ell+1, q_\ell))$ as j takes all values in the range $\{1, 2, ..., \ell-1\}$ such that the repunit $U_{n_j}(q_j+1, q_j)$ contains an extra prime divisor.

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